

Stationary compressible Navier Stokes Equations with inflow condition in domains with piecewise analytical boundaries

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MSC: 35Q30, 76N10

Keywords: Compressible Navier-Stokes equations, slip boundary condition, inflow boundary condition, strong solutions

Abstract

We show existence of strong solutions in Sobolev-Slobodetskii spaces to the stationary compressible Navier-Stokes equations with inflow boundary condition. Our result holds provided certain condition on the shape of the boundary around the points where characteristics of the continuity equation are tangent to the boundary, which holds in particular for piecewise analytical boundaries.

1 Introduction

We investigate the existence of regular solutions to stationary barotropic compressible Navier-Stokes equations in a two dimensional bounded domain Ω with nonzero inflow/outflow through the boundary. We show existence of a solution in fractional Sobolev spaces $u \in W_p^{1+s}(\Omega)$, $\rho \in W_p^s(\Omega)$, where u is the velocity field of the fluid and ρ is the density. Our choice of functional spaces allows to overcome the problem of singularity in the continuity equation and obtain boundedness of the density. Before we formulate the problem more

precisely we give a brief overview of the state of art in the topic, focusing on the scope of interest of this paper, that is on regular stationary solutions, mentioning also the most important results concerning global weak stationary solutions.

The mathematical theory of stationary solutions to the Navier-Stokes equations describing compressible flows started to develop in early 80's with certain results on the existence of regular solutions, first in Hilbert spaces and later in L_p framework ([2]). However, all of these results required certain smallness assumptions on the data and concerned mostly homogeneous boundary conditions with vanishing normal component of the velocity.

In the 90's the famous result of Lions [6] on the existence of weak solutions for homogeneous Dirichlet boundary conditions triggered the development of global existence theory of weak solutions. The result was improved by Feireisl [3] and then extended by Mucha and Pokorný ([8],[17]) to the case of slip boundary conditions. Certain improvements in the theory of regular solutions has been made in the early nineties ([10],[11]) but mostly for homogeneous boundary data.

It should be emphasized that all above mentioned global results concern the case of normal component of the velocity vanishing on the boundary. If the normal component of the velocity does not vanish, the hyperbolicity of the continuity equation makes it necessary to prescribe the density on the part of the boundary where the fluid enters the domain, called briefly the inflow part. We refer to such problem as an inflow problem. Such problems, the mathematical understanding of which is extremely important from the point of view of applications, are so far behind the scope of the global existence theory. Nontrivial boundary terms yields impossible to get basic a priori estimates and further problems are encountered with the issue of existence and uniqueness for the continuity equation.

The mathematical investigation of inflow/outflow problems began with the work of Valli and Zajăczkowski [18], who investigated time-dependent problem obtaining also an existence result in the stationary case. Then the development of existence theory for inhomogeneous boundary data has been hindered by mathematical difficulties on the one hand and the interest turned mostly towards global existence of weak solutions on the other, until the work by Kweon and Kellogg [4]. More recently, the existence theory has been developed motivated by applications in shape optimization by Plotnikov and Sokolowski ([15] and the monograph [16]). At first glance a natural functional space for regular solutions is W_p^1 for the density and

W_p^2 for the velocity. A regular solution is then understood as a function with weak derivatives satisfying the equations almost everywhere. However, except some special classes of domains we are not able to obtain the solutions in the above class for arbitrarily large p (see [4] and [15] - note that the limitation on p in both cited papers, although formulated in a different way is indeed the same, what is a strong evidence of its optimality). The reason is the singularity arising in the solution of steady transport equation around the points where characteristics of this hyperbolic equation become tangent to the boundary, we refer to these points as singularity points.

On the other hand, the range $p > n$ is important since it gives boundedness of the density due to the imbedding theorem. The results from [4] and [15] cover a part of this range, namely $2 < p < 3$. However, further increase of p is impossible even under relaxation of the boundary singularity. Further investigation of this singularity is therefore an interesting question in view of the development of the theory of regular solutions.

One possible way to obtain existence for $\infty > p > n$ is to investigate some special domains, such as a cylindrical domain in [13] for barotropic case and [14] for system with thermal effects or an unbounded domain in [5]. A possible way to overcome the singularity problem described above in a general domain is an appropriate choice of functional spaces. In this paper we use fractional Sobolev spaces W_p^s equipped with Sobolev-Slobodetskii norm defined below. Our analysis shows that we are able to show existence of the solutions for $sp > n$ which gives boundedness of the density. We need to impose a certain limitation on the boundary around the singularity points, however this assumption is weaker than in [4] and [15] and turns out quite natural, in particular it is satisfied by analytical boundaries.

Our result, which is to our knowledge first in the framework of fractional spaces in the stationary case, shows that this choice of functional spaces is in a sense natural for this problem and therefore is not only of purely mathematical interest. In particular it may indicate a possible direction for the development of the theory of global existence which, as it has been mentioned above, is still unavailable for inflow/outflow problems.

For more complete overview of known results in the mathematical theory of compressible flows we refer to the monographs [12] and [16].

Let us move to a precise statement of the problem under consideration. We investigate stationary flow of a barotropic fluid in a two dimensional, bounded domain. The system is supplied with inhomogeneous slip boundary conditions on the velocity. In particular, the normal component of the

velocity does not vanish and, as explained above, we have to prescribe the density on the part of the boundary where the flow enters the domain. The complete system reads

$$\begin{aligned}
\rho v \cdot \nabla v - \mu \Delta v - (\mu + \nu) \nabla \operatorname{div} v + \nabla \pi(\rho) &= 0 && \text{in } \Omega, \\
\operatorname{div}(\rho v) &= 0 && \text{in } \Omega, \\
n \cdot 2\mu \mathbf{D}(u) \cdot \tau_k + f v \cdot \tau_k &= b, \quad k = 1, 2 && \text{on } \Gamma, \\
n \cdot v &= d && \text{on } \Gamma, \\
\rho &= \rho_{in} && \text{on } \Gamma_{in},
\end{aligned} \tag{1}$$

where the velocity field of the fluid v and the density ρ are the unknown functions describing the flow. We distinguish the parts of the boundary: $\Gamma_{in} = \{x \in \Gamma : v \cdot n < 0\}$, $\Gamma_{out} = \{x \in \Gamma : v \cdot n > 0\}$, $\Gamma_0 = \{x \in \Gamma : v \cdot n = 0\}$.

Our goal is to show existence of strong solution $(u, \rho) \in W_p^{1+s} \times W_p^s$ to the system (4), where $s > \frac{n}{p}$ (for the sake of completeness we recall the definition of Sobolev-Slobodetskii spaces and their basic properties below). Our motivation for the choice of this fractional order space for the density has been explained above; we want to solve the problem of singularity in the solution of the continuity equation around the singularity points. On the other hand, by the imbedding theorem we have $W_p^s(\Omega) \in L_\infty(\Omega)$ (see Lemma 1 below). Then the choice of the space W_p^{1+s} for the velocity follows naturally from the structure of (1).

We are interested in a solution to (4) which is close to the constant flow

$$(\bar{v}, \bar{\rho}) \equiv ([1, 0], 1).$$

Our method works for a wider class of solutions in which x_1 is in a sense dominating direction.

In order to formulate our main result let us introduce the following quantity to measure the distance of the data of the problem (1) from $(\bar{v}, \bar{\rho})$.

$$D_0 = \|b - f\tau^{(1)}\|_{W_p^{s-1/p}(\Gamma)} + \|d - n^{(1)}\|_{W_p^{1+s-1/p}(\Gamma)} + \|\rho_{in} - 1\|_{W_p^s(\Gamma_{in})}. \tag{2}$$

We are now in a position to formulate our main result.

Theorem 1. *Assume that D_0 defined in (2) is small enough, where s is sufficiently small and $sp > 2$. Let f be large enough on Γ_{in} . Then there exists a solution $(v, \rho) \in W_p^{1+s} \times W_p^s$ to the system (1) such that*

$$\|v - \bar{v}\|_{W_p^{1+s}} + \|\rho - \bar{\rho}\|_{W_p^s} \leq E(D_0), \tag{3}$$

where $E(\cdot)$ is a Lipschitz function. This solution is unique in the class of solutions satisfying (3).

The rest of the paper is organized as follows. In the remaining of the present section we reformulate the problem (1) introducing perturbations as new unknowns, obtaining system (4). Then we recall basic properties of the functional spaces we use. In Section 2 we introduce linearization of (4) and show a priori estimates, in particular we deal with steady transport equation obtained from linearization of the continuity equation. We show W_p^s estimate which is crucial in showing the estimate for linearization of the original problem in desired space. The rest of the estimate is standard applying elliptic regularity results and properties of Helmholtz decomposition. The details will be presented in forthcoming full version of the paper.

To remove inhomogeneity from the boundary condition (4)₄, by Extension Theorem ([9]) we can construct $u_0 \in W_p^{1+s}$ such that

$$u_0 \cdot n|_\Gamma = d - n^{(1)}, \quad \|u_0\|_{W_p^{1+s}} \leq C \|d - n^{(1)}\|_{W_p^{1+s-1/p}(\Gamma)}.$$

Introducing the perturbations

$$u = v - \bar{v} - u_0 \quad \text{and} \quad w = \rho - \bar{\rho}$$

we obtain the system

$$\begin{aligned} \partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u + \gamma \nabla w &= F(u, w) & \text{in } \Omega, \\ \operatorname{div} u + \partial_{x_1} w + (u + u_0) \cdot \nabla w &= G(u, w) & \text{in } \Omega, \\ n \cdot 2\mu \mathbf{D}(u) \cdot \tau + f u \cdot \tau &= B & \text{on } \Gamma, \\ n \cdot u &= 0 & \text{on } \Gamma, \\ w &= w_{in} & \text{on } \Gamma_{in}, \end{aligned} \tag{4}$$

where $\gamma = \pi'(1)$ and

$$\begin{aligned} F(u, w) &= -w(u + \bar{v} + u_0) \cdot \nabla(u + u_0) - (u_0 \cdot \nabla u) - u \cdot \nabla u_0 \\ &\quad + \mu \Delta u_0 + (\nu + \mu) \nabla \operatorname{div} u_0 - u_0 \cdot \nabla u_0 - [\pi'(w + w_0 + 1) - \pi'(1)] \nabla w, \\ G(u, w) &= -(w + 1) \operatorname{div} u_0 - w \operatorname{div} u \end{aligned} \tag{5}$$

and

$$B_i = b - 2\mu n \cdot \mathbf{D}(u_0) \cdot \tau_i - f \tau_i^{(1)}.$$

It can be seen easily that

$$\begin{aligned} \|F\|_{L^2} &\leq C[(\|u_0\|_{W_p^{1+s}} + \|w_0\|_{W_p^s}) + (\|u\|_{W_p^{1+s}} + \|w\|_{W_p^s})^2], \\ \|G\|_{L^2} &\leq C[(\|u_0\|_{W_p^{1+s}} + \|w_0\|_{W_p^s}) + (\|u\|_{W_p^{1+s}} + \|w\|_{W_p^s})^2]. \end{aligned} \tag{6}$$

From now on we focus on the system (4). Our goal is to show existence of a solution $(u, w) \in W_p^{1+s} \times W_p^s$ for given small functions $u_0 \in W_p^{1+s}$ and $w_0 \in W_p^s$.

1.1 Functional spaces.

We use standard Sobolev spaces W_p^k with natural k , which consist of functions with the weak derivatives up to order k in $L_p(\Omega)$, for the definition we refer for example to [1]. However, most important for our result are Sobolev-Slobodetskii spaces W_p^s with fractional s . For the sake of completeness we recall the definition here. By $W_p^s(\Omega)$ we denote the space of functions for which the norm:

$$\|f\|_{W_p^s(\Omega)} = \left(\iint_{\Omega^2} \frac{|f(x) - f(y)|^p}{|x - y|^{2+sp}} dx dy \right)^{1/p} \quad (7)$$

is finite.

Let us recall two important features of Sobolev-Slobodetskii spaces. We formulate it in a simplified way convenient for our applications.

Lemma 1. (*Imbedding Theorem*). *Let $u \in W_p^s$ with $sp > n$. Then for $q \leq \infty$*

$$\|u\|_{L_q} \leq C(q, \Omega) \|u\|_{W_p^s}. \quad (8)$$

Lemma 2. (*Interpolation Inequality*). *Let $u \in W_p^s$ with $sp > n$. Then for $q \leq \infty$, for any $\delta > 0$*

$$\|u\|_{L_q} \leq \delta \|u\|_{W_p^s} + C(\delta) \|u\|_{L_2}. \quad (9)$$

Finally, let us denote

$$V = \{v \in W_2^1(\Omega) : v \cdot n|_{\Gamma} = 0\}. \quad (10)$$

1.2 The domain.

We consider a two dimensional bounded domain with sufficiently smooth boundary where inflow and outflow parts of the boundary are given piecewise by functions $x_1(x_2)$. More precisely, we assume that there exists $N_1, N_2, k_1, \dots, k_{N_1+1}$ and l_1, \dots, l_{N_2+1} such that

$$\Gamma_{in} = \Gamma_{in}^1 \cup \dots \cup \Gamma_{in}^{N_1} \quad \text{and} \quad \Gamma_{out} = \Gamma_{out}^1 \cup \dots \cup \Gamma_{out}^{N_2}, \quad (11)$$

where

$$\Gamma_{in}^i = \underline{x}_1^i(x_2), \quad x_2 \in (k_i, k_{i+1})$$

and

$$\Gamma_{out}^i = \overline{x}_1^i(x_2), \quad x_2 \in (l_i, l_{i+1}).$$

Finally we assume

$$\underline{x}_1^i(k_{i+1}) \geq \underline{x}_1^{i+1}(k_{i+1})$$

and

$$\overline{x}_1^i(l_{i+1}) \geq \overline{x}_1^{i+1}(l_{i+1}).$$

Hence Γ_0 consists of a set of points $(\underline{x}_1^i(k_{i+1}), k_{i+1})$, $(\overline{x}_1^i(l_{i+1}), l_{i+1})$ or straight lines $[\underline{x}_1^{i+1}(k_{i+1}), \underline{x}_1^i(k_{i+1})] \times \{k_{i+1}\}$ and $[\overline{x}_1^{i+1}(l_{i+1}), \overline{x}_1^i(l_{i+1})] \times \{l_{i+1}\}$. It is well known and was already mentioned in the introduction that existence of regular solutions to inflow problem (1) requires certain assumptions on the shape of the boundary around the singularity points. We also need an assumption of this kind, in order to formulate it notice that around each singularity point the boundary is given as a function $x_2(x_1)$ (in general we have different functions x_2^i but now we focus on one singularity point). This function can be constant in some neighborhood of the singularity point at most at one side (in x_1 direction) of this singularity point. We assume that whenever it is not constant, it satisfies

$$\exists N \in \mathbb{N} : |x_2(x_1) - x_2(y_1)| \geq C|x_1 - y_1|^N. \quad (12)$$

This condition is weaker than in ([4] and [15]) and means that the boundary around the singularity points is less flat than some polynomial. It seems quite technical in the above formulation but in fact it is satisfied by a wide class of functions, in particular by piecewise analytical boundaries what is shown in the following lemma:

Lemma 3. *Assume that x_2 is an analytic function of x_1 around the singularity points. Then (12) holds.*

Proof. It is enough to show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is analytic in some $[-r, r]$, $f(0) = 0$ and $f \neq 0$ then

$$|f(x)| \geq C|x|^N \quad \text{for } x \in (-l, l) \quad (13)$$

for some $C, N > 0$ and $l < r$ sufficiently small. Since $f \neq 0$ and f is analytic, we must have $f^{(n)}(0) \neq 0$ for some n . Let $f^{(k)}(0)$ be the first derivative not vanishing in 0. Then we have

$$f(x) = \frac{f^{(k)}(0)}{k!}x^k + R^{k+1}(x),$$

where $|R^{k+1}(x)| \leq M|x|^{k+1}$ for $x \in (-r, r)$. Hence

$$|f(x)| \geq \left| \frac{f^{(k)}(0)}{k!} \right| |x|^k - M|x|^{k+1} = \left(\frac{f^{(k)}(0)}{k!} - M|x| \right) |x|^k. \square$$

Since our proofs require accurate treatment of different parts of the boundary, in particular in the neighborhood of the singularity points, conducting them in a full generality described above would lead to numerous and unnecessary complications. However we can show the proofs for a simple and representative domain with two singularity points and later explain briefly how they are generalized to the class of domains defined above.

The domain. Simple representative case. For simplicity consider the case with $N_1 = N_2 = 1$, $k_1 = l_1 = 0$, $k_2 = l_2 = b > 0$ and

$$\underline{x}_1(0) = \overline{x}_1(0) = \underline{x}_1(b) = \overline{x}_1(b) = 0.$$

Hence we have only two singularity points $(0, 0)$ and $(0, b)$ and inflow and outflow parts are given as $\Gamma_{in} = (\underline{x}_1(x_2), x_2)$ and $\Gamma_{out} = (\overline{x}_1(x_2), x_2)$, $x_2 \in (0, b)$. Around the singularity points the boundary is given as a graph $x_2(x_1)$. We assume that it satisfies the condition (12) which can be rewritten as

$$\exists \delta > 0 : |\overline{x}_1(x_2) - \overline{x}_1(y_2)| + |\underline{x}_1(x_2) - \underline{x}_1(y_2)| \leq C|x_2 - y_2|^\delta. \quad (14)$$

Let us make the following assumption relating the boundary data and the boundary around the singularity points:

Assumption 1: $(d-n^{(1)})\underline{x}_1'(x_2)$ and $(d-n^{(1)})\overline{x}_1'(x_2)$ are bounded around the singularity points.

2 Linearization and priori bounds

In this section we derive *a priori* estimates for the following linearization of system (4)

$$\begin{aligned}
\partial_{x_1} u - \mu \Delta u - (\nu + \mu) \nabla \operatorname{div} u + \gamma \nabla w &= F & \text{in } \Omega, \\
\operatorname{div} u + \partial_{x_1} w + U \cdot \nabla w &= G & \text{in } \Omega, \\
n \cdot 2\mu \mathbf{D}(u) \cdot \tau + f u \cdot \tau &= B & \text{on } \Gamma, \\
n \cdot u &= 0 & \text{on } \Gamma, \\
w &= w_{in} & \text{on } \Gamma_{in},
\end{aligned} \tag{15}$$

where $U \in W_p^{1+s}$ is small and satisfies $U \cdot n|_\Gamma = d - n^{(1)}$.

2.1 Energy estimate

In this section we show energy estimate for the solutions of (15).

Lemma 4. *Let (u, w) be a sufficiently smooth solution to system (15) with given functions $(F, G, B) \in (V^* \times L_2 \times L_2(\Gamma))$. Then*

$$\|u\|_{W_2^1} + \|w\|_{L_2} \leq C[\|F\|_{V^*} + \|G\|_{L_2} + \|B\|_{L^2(\partial\Omega)}], \tag{16}$$

where C is independent from the boundary data and V^* is a dual space to V defined in (10).

In order to show (16) we apply a standard energy method. Multiplying the first equation of (15) by u and integrating over Ω we get using the boundary condition (15)₃:

$$\begin{aligned}
\int_\Omega 2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u dx + \int_{\partial\Omega} (f + \frac{n^{(1)}}{2}) u^2 d\sigma + \int_\Omega \gamma \nabla w u dx = \\
- \int_\Omega \partial_{x_1} u u dx + \int_\Omega F u dx + \int_{\partial\Omega} B(u \cdot \tau) d\sigma
\end{aligned} \tag{17}$$

The boundary term on the lhs will be positive for $f \geq 0$ on Γ_{out} and $f \geq \frac{n^{(1)}}{2}$ on Γ_{in} . Next we integrate by parts the last term of the l.h.s of (17). Using (15)₂ we obtain:

$$\begin{aligned}
\gamma \int_\Omega \nabla w u dx &= -\gamma \int_\Omega \operatorname{div} u w dx = \gamma \int_\Omega (\partial_{x_1} w w + U \cdot \nabla w w - G w) dx \\
&= \gamma \left[\frac{1}{2} \int_\Gamma w^2 n_1 d\sigma - \frac{1}{2} \int_\Omega w^2 \operatorname{div} U - \int_\Omega G w dx \right].
\end{aligned}$$

We will also use the following Korn inequality:

$$\int_{\Omega} 2\mu \mathbf{D}^2(u) + \nu \operatorname{div}^2 u \, dx \geq C_K \|u\|_{W_2^1}^2, \quad (18)$$

where $C_K = C_K(\Omega)$. Using (17), (??) and (18) we get:

$$\begin{aligned} & \|u\|_{W_2^1}^2 + \int_{\Gamma_{out}} w^2 n_1 \, d\sigma \leq \\ & \leq C \left[\int_{\Omega} w^2 \operatorname{div} U \, dx + \gamma \int_{\Omega} G w \, dx + \int_{\Omega} F u \, dx + \int_{\Gamma} B(u \cdot \tau) \, d\sigma + \int_{\Gamma_{in}} w_{in}^2 n^{(1)} \, d\sigma \right]. \end{aligned}$$

Next, using Holder and Young inequalities, the fact that $w^2 n^{(1)} > 0$ on Γ_{out} and the trace theorem to the boundary term we get for any $\delta > 0$:

$$\|u\|_{W_2^1}^2 \leq (\delta + \|U\|_{W_p^{1+s}}) \|w\|_{L_2}^2 + C(\delta) \|G\|_{L_2}^2 + C \left[(\|F\|_{V^*} + \|B\|_{L_2(\Gamma)}) \|u\|_{W_2^1} \right],$$

which yields

$$\|u\|_{W_2^1} \leq (\delta + \|U\|_{W_p^{1+s}}) \|w\|_{L_2} + C(\delta) \|G\|_{L_2} + C(\|F\|_{V^*} + \|B\|_{L_2(\Gamma)}). \quad (19)$$

To estimate the first term of the r.h.s. we find a bound on $\|w\|_{L_2}$. Let us define:

$$a = \min \{ \underline{x}_1(x_2) : x_2 \in (0, b) \}$$

From (15)₂ we have

$$\partial_{x_1} w + U \cdot \nabla w = G - \operatorname{div} u$$

In order to estimate $\|w\|_{L_2}$, for $x \in \Omega$ let us denote by γ_x a characteristic of the operator $\partial_{x_1} + U \cdot \nabla$ passing by x , and by \underline{x} intersection of γ_x with Γ_{in} . Due to regularity and smallness of U γ_x is close to a straight line $\{x_2 = c\}$. Now we can write

$$w(x) = w_{in}(\underline{x}) + \int_{\gamma_x} (G - \operatorname{div} u) \, dl_{\gamma}. \quad (20)$$

By Jensen inequality we have

$$\left(\int_{\gamma} G - \operatorname{div} u \, dl_{\gamma} \right)^2 \leq |\gamma| \int_{\gamma} |G|^2 + |\operatorname{div} u|^2 \, dl_{\gamma}.$$

Hence applying (20) we get

$$\|w\|_{L_2} \leq C(\Omega) \left(\|w_{in}\|_{L_2(\Gamma_{in})} + \|G\|_{L_2} + \|u\|_{W_2^1} \right). \quad (21)$$

Now we combine (19) and (20). By smallness of U we can fix δ in (19) small enough to put the term $\delta + \|U\|_{W_p^{1+s}}$ on the left obtaining (16), which completes the proof of the lemma. \square

2.2 Steady transport equation

In this section we show W_p^s estimate for the steady transport equation (22), which is a crucial step in showing W_p^s estimate for (15).

Lemma 5. *Let*

$$w + w_{x_1} + u \cdot \nabla w = H, \quad w|_{\Gamma_{in}} = w_{in} \quad (22)$$

with $u \in W_p^{1+s}$ small, $H \in W_p^s$ and $w_{in} \in W_p^s(\Gamma_{in})$. Assume that (12) holds and the boundary data satisfy following additional assumptions:

$$(u \cdot n) \overline{x_1}(x_2) \text{ and } (u \cdot n) \underline{x_1}(x_2) \text{ are bounded around the singularity points.} \quad (23)$$

and

$$w_{in} = 0 \quad \text{in a neighbourhood of singularity points.} \quad (24)$$

Then

$$\|w\|_{W_p^s} \leq C[\|H\|_{W_p^s} + \|w\|_{L_2}] \quad (25)$$

where $C = C(s, p, \Omega)$.

Recalling the definition of Sobolev-Slobodetskii norm we write (22) in x and y . Using identities of a kind of $\nabla_x w(y) = 0$ we can write

$$w(x) + \partial_{x_1}[w(x) - w(y)] + u(x) \cdot \nabla_x[w(x) - w(y)] = H(x)$$

$$w(y) + \partial_{y_1}[w(y) - w(x)] + u(y) \cdot \nabla_y[w(y) - w(x)] = H(y)$$

We multiply first equation by $\frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{\phi_\epsilon(x,y)}$ and second by $\frac{|w(x)-w(y)|^{p-2}(w(y)-w(x))}{\phi_\epsilon(x,y)}$, where

$$\phi_\epsilon(x, y) = \epsilon + |x - y|^{2+sp}. \quad (26)$$

Then we add the equations and perform $\iint_{\Omega^2} dx dy$. Since

$$w(x)(w(x) - w(y)) + w(y)(w(y) - w(x)) = [w(x) - w(y)]^2$$

and

$$H(x)(w(x) - w(y)) + H(y)(w(y) - w(x)) = (H(x) - H(y))(w(x) - w(y)),$$

we obtain on the left hand side

$$\iint_{\Omega^2} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy \xrightarrow{\epsilon \rightarrow 0} \|w\|_{W_p^s}^p. \quad (27)$$

On the r.h.s we have using Holder inequality:

$$\begin{aligned} & \iint_{\Omega^2} \frac{(H(x) - H(y))|w(x) - w(y)|^{p-2}(w(x) - w(y))}{\phi_\epsilon(x, y)} \leq \\ & \left(\iint_{\Omega^2} \frac{|H(x) - H(y)|^p}{\phi_\epsilon(x, y)} \right)^{1/p} \left(\iint_{\Omega^2} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right)^{1-1/p} \\ & \xrightarrow{\epsilon \rightarrow 0} \|H\|_{W_p^s} \|w\|_{W_p^s}^{p-1}. \end{aligned} \quad (28)$$

Therefore at this stage we obtain

$$\begin{aligned} & \|w\|_{W_p^s} + \frac{1}{p} \iint_{\Omega^2} \frac{\partial_{x_1}|w(x) - w(y)|^p + \partial_{y_1}|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy \\ & + \frac{1}{p} \iint_{\Omega^2} \frac{u(x) \cdot \nabla_x |w(x) - w(y)|^p + u(y) \cdot \nabla_y |w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy \\ & \leq \|H\|_{W_p^s} \|w\|_{W_p^s}^{p-1} \end{aligned} \quad (29)$$

We wrote the above intermediate step down to fix the attention and show where $\|w\|_{W_p^s}$ appears in a natural way. The rest of the proof consist in dealing with the integral terms in (29). This is where all the difficulties are hidden and our assumptions on the boundary and boundary data will come into play. Since we want to have W_p^s norm, we have to get rid of the derivatives of w and this is done obviously integrating by parts. Let us start with the first integral term. We have

$$\frac{\partial_{x_1}|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} = \partial_{x_1} \left(\frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right) - |w(x) - w(y)|^p \partial_{x_1} \left(\frac{1}{\phi_\epsilon(x, y)} \right). \quad (30)$$

By the definition of $\phi_\epsilon(x, y)$ we have

$$\partial_{x_i} \frac{1}{\phi_\epsilon(x, y)} = - \frac{(2 + sp) |x - y|^{sp} (x_i - y_i)}{\phi_\epsilon^2} = - \partial_{y_i} \frac{1}{\phi_\epsilon(x, y)}.$$

In particular,

$$\nabla_x \phi_\epsilon(x, y) = -\nabla_y \phi_\epsilon(x, y). \quad (31)$$

Using (30) we get

$$\begin{aligned} & \iint_{\Omega^2} \left\{ \frac{(w(x)-w(y))|w(x)-w(y)|^{p-2} \partial_{x_1}[w(x)-w(y)]}{\phi_\epsilon(x, y)} \right. \\ & \quad \left. + \frac{(w(y)-w(x))|w(x)-w(y)|^{p-2} \partial_{y_1}[w(y)-w(x)]}{\phi_\epsilon(x, y)} \right\} dx dy = \\ & = \frac{1}{p} \iint_{\Omega^2} \left\{ \partial_{x_1} \left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x, y)} \right) + \partial_{y_1} \left(\frac{|w(x)-w(y)|^p}{\phi_\epsilon(x, y)} \right) \right\} dx dy \\ & - \frac{1}{p} \iint_{\Omega^2} \left\{ |w(x)-w(y)|^p \partial_{x_1} \left(\frac{1}{\phi_\epsilon(x, y)} \right) + |w(x)-w(y)|^p \partial_{y_1} \left(\frac{1}{\phi_\epsilon(x, y)} \right) \right\} dx dy. \end{aligned} \quad (32)$$

Taking into account (31), the integrand in the second integral on the rhs of (32) vanishes identically. Combining (31) with the identities

$$\nabla_x |w(x) - w(y)|^p = p|w(x) - w(y)|^{p-2} (w(x) - w(y)) \nabla_x w(x) \quad (33)$$

and

$$\nabla_y |w(x) - w(y)|^p = -p|w(x) - w(y)|^{p-2} (w(x) - w(y)) \nabla_y w(y) \quad (34)$$

we see that the first integral on the rhs of (32) adds up to

$$\frac{2}{p} \iint_{\Omega^2} \partial_{x_1} \left(\frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right) dx dy = \frac{2}{p} \int_{\Omega} \int_{\Gamma} \frac{|w(x) - w(y)|^p n^{(1)}}{\phi_\epsilon(x, y)} dS(x) dy. \quad (35)$$

Now by (64), (35) can be rewritten as

$$\int_{\Omega} dy \int_0^b \left[\frac{|w(\overline{x_1}(x_2), x_2) - w(y)|^p}{\phi_\epsilon(\overline{x_1}(x_2), x_2, y)} - \frac{|w(\underline{x_1}(x_2), x_2) - w(y)|^p}{\phi_\epsilon(\underline{x_1}(x_2), x_2, y)} \right] dx_2. \quad (36)$$

Now we focus on (36). First of all notice that Γ_{out} part will be positive, hence can be omitted in the estimate. However, the Γ_{in} part give rise to some problems when we pass with $\epsilon \rightarrow 0$ when y is close to Γ_{in} . Hence it is useful to define, for given $\eta > 0$,

$$\Omega_{in}^\eta := \{y \in \Omega : \text{dist}(y, \Gamma_{in}) < \eta\}. \quad (37)$$

For simplicity we now skip η and write Ω_{in} . Let us denote $\Omega_{in} = \Omega_{in}^r \cup \Omega_{in}^s$ where Ω_{in}^s is the neighborhood of the singularity points. Now write (36) as

$$\int_{\Omega} (\dots) dy = \int_{\Omega_{in}^r} (\dots) dy + \int_{\Omega_{in}^s} (\dots) dy + \int_{\Omega \setminus \Omega_{in}} (\dots) dy =: I_1^r + I_1^s + I_2,$$

where

$$(\dots) = \int_0^b -\frac{|w(\underline{x}_1(x_2), x_2) - w(y)|^p}{\phi_\epsilon(\underline{x}_1(x_2), x_2, y)} dx_2.$$

The I_2 integral is straightforward as we have $|(\underline{x}_1(x_2), x_2) - y| > \eta$ and so

$$I_2 \leq C(\eta)[\|w_{in}\|_{L_p(\Gamma_{in})} + \|w\|_{L_p}]^p. \quad (38)$$

The I_1 term is more involved. For $y = (y_1, y_2)$ let us denote by γ_y a characteristic curve of the operator $(\partial_{x_1} + u \cdot \nabla)$ connecting y with Γ_{in} . Let us denote the beginning of γ_y (lying on Γ_{in}) by \underline{y} . By smallness of $\|u\|_{W_p^{1+s}}$, γ_y is close to straight line and in particular

$$|\gamma_y| \simeq |y_1 - \underline{x}_1(y_2)| \simeq |\underline{y} - y|. \quad (39)$$

Now let $\underline{x} = (\underline{x}_1(x_2), x_2) \in \Gamma_{in}$. Then on Ω_{in}^r we have either

$$|\underline{x} - y| \sim |\underline{y} - y| \sim |\underline{x} - \underline{y}|, \quad (40)$$

$$|\underline{x} - y| \sim |\underline{x} - \underline{y}| \quad \text{and} \quad |\underline{y} - y| << |\underline{x} - y| \quad (41)$$

or

$$|\underline{x} - \underline{y}| << |\underline{y} - y| \sim |\underline{x} - y|. \quad (42)$$

In either case we can write

$$\frac{|w(\underline{x}) - w(y)|^p}{|\underline{x} - y|^{2+sp}} \leq \frac{|w(\underline{x}) - w(\underline{y})|^p}{|\underline{x} - y|^{2+sp}} + \frac{|w(\underline{y}) - w(y)|^p}{|\underline{x} - y|^{2+sp}}. \quad (43)$$

With the first term on the rhs of (43) we have

$$\begin{aligned} \int_{\Omega_{in}^r} dy \int_0^b \frac{|w(\underline{x}) - w(y)|^p}{|\underline{x} - y|^{2+sp}} dx_2 &\sim \int_{J^r} dy_2 \int_0^b \frac{|w(\underline{x}) - w(y)|^p}{|\underline{x} - y|^{1+sp}} dx_2 \leq \\ &\leq C \int_{J^r} dy_2 \int_0^b \frac{|w(\underline{x}) - w(y)|^p}{|\underline{x} - y|^{1+sp}} dx_2 \leq C \|w_{in}\|_{W_p^s(\Gamma_{in})}, \end{aligned} \quad (44)$$

where J^r denotes a projection of Ω_{in}^r on x_2 . Here we used the fact that $|\underline{x} - \underline{y}| \leq C|\underline{x} - y|$ resulting from (40)-(42). In the last step we identified dx_2 with measure on Γ_{in} , since except neighbourhood of singularity points these are equivalent by (65). Notice that in this part of the estimate the norm of boundary data appears naturally.

In the second term on the rhs of (43) we express $w(y) - w(\underline{y})$ by the integral along γ_y using the continuity equation. Applying Jensen inequality we get

$$\begin{aligned}
\int_{\Omega_{in}^r} dy \int_0^b \frac{|w(y) - w(\underline{y})|^p}{|\underline{x} - y|^{2+sp}} dx_2 &\leq \int_{\Omega_{in}^r} dy \int_0^b \frac{|\int_{\gamma_y} (H - w) dl_{\gamma_y}|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq \\
&(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega_{in}^r} dy \int_0^b \frac{|\gamma_y|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq \\
C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega_{in}^r} dy \int_0^b \frac{|\underline{y} - y|^p}{|\underline{x} - y|^{2+sp}} dx_2 &\leq \\
C(\|H\|_{L_\infty} + \|w\|_{L_\infty})^p \int_{\Omega_{in}^r} dy \int_0^b \frac{|\underline{x} - y|^p}{|\underline{x} - y|^{2+sp}} dx_2, &
\end{aligned} \tag{45}$$

where we used sequentially (39) and (40)-(42) which yields in particular $|\underline{y} - y| \leq C|\underline{x} - y|$. The last integral is finite for $s < 1$. Combining (43), (44) and (45) we get

$$I_1^r \leq C(\|H\|_{L_\infty} + \|w\|_{L_\infty} + \|w_{in}\|_{W_p^s(\Gamma_{in})})^p. \tag{46}$$

Around the singularity points we can proceed in a similar way but this time we can have

$$|\underline{x} - y| \ll |\underline{x} - \underline{y}| \sim |\underline{y} - y|. \tag{47}$$

Hence we cannot repeat directly (44) and (44). We have to control $|\underline{x} - \underline{y}|$ and here we use the assumption (12). More precisely, let us define by \tilde{y} the intersection of a line $\{(\underline{x}_1(x_2), t) : t \in \mathbb{R}\}$ with a characteristic γ_y connecting Γ_{in} with y . Let us denote by $\gamma_{\tilde{y}}$ the part of the characteristic connecting Γ_{in} with \tilde{y} . Then by smallness of u we have

$$|\gamma_{\tilde{y}}| \sim |\underline{x}_1(x_2) - \underline{x}_1(y_2)| \tag{48}$$

and

$$|\tilde{y} - y| \sim |\underline{x}_1(x_2) - y_1|.$$

Now we can have either

$$|\underline{x} - \tilde{y}| \sim |\underline{y} - \tilde{y}| \sim |\underline{x} - y| \tag{49}$$

or

$$|\underline{x} - \tilde{y}| \ll |\underline{y} - \tilde{y}| \sim |\underline{x} - y|. \tag{50}$$

Hence we can repeat (43) but with \tilde{y} instead of \underline{y} :

$$\frac{|w(\underline{x}) - w(y)|^p}{|\underline{x} - y|^{2+sp}} \leq \frac{|w(\underline{x}) - w(\underline{y})|^p}{|\underline{x} - y|^{2+sp}} + \frac{|w(\underline{y}) - w(\tilde{y})|^p}{|\underline{x} - y|^{2+sp}} + \frac{|w(\tilde{y}) - w(y)|^p}{|\underline{x} - y|^{2+sp}}. \tag{51}$$

The first term on the rhs of (51) vanishes due to the assumption (24). We have to deal with two remaining terms. Similarly as before, in both terms we replace the value of w with integral along γ_y . The last term is analogous to I_1^r and we get

$$\int_{\Omega_{in}^s} dy \int_0^b \frac{|w(\tilde{y}) - w(y)|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq C(||H||_{L_\infty} + ||w||_{L_\infty})^p \quad \text{for } s < 1. \quad (52)$$

In the second term we have $w(\tilde{y}) - w(\underline{y}) = \int_{\gamma_{\tilde{y}}} (H - w) dl_{\gamma_{\tilde{y}}}$. Now we can write

$$\begin{aligned} \int_{\Omega_{in}^s} dy \int_0^b \frac{|w(\underline{y}) - w(\tilde{y})|^p}{|\underline{x} - y|^{2+sp}} &\leq \int_{\Omega_{in}^s} dy \int_0^b \frac{|\int_{\gamma_{\tilde{y}}} (H - w) dl_{\gamma_{\tilde{y}}}|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq \\ & (||H||_{L_\infty} + ||w||_{L_\infty})^p \int_{\Omega_{in}^s} dy \int_0^b \frac{|\gamma_{\tilde{y}}|^p}{|\underline{x} - y|^{2+sp}} \leq \\ & C(||H||_{L_\infty} + ||w||_{L_\infty})^p \int_{\Omega_{in}^s} dy \int_0^b \frac{|\underline{x}_1(x_2) - \underline{x}_1(y_2)|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq \\ & C(||H||_{L_\infty} + ||w||_{L_\infty})^p \int_{\Omega_{in}^s} dy \int_0^b |x_2 - y_2|^{\delta p - 2 - sp} dx_2. \end{aligned}$$

In the above series of inequalities we used Jensen inequality, (48) and (14). The last integral is finite for $s < \delta$ and we conclude

$$\int_{\Omega_{in}^s} dy \int_0^b \frac{|w(\underline{y}) - w(\tilde{y})|^p}{|\underline{x} - y|^{2+sp}} dx_2 \leq C(||H||_{L_\infty} + ||w||_{L_\infty})^p \quad \text{for } s < 1. \quad (53)$$

Combining (52) and (53) we conclude

$$I_1^s \leq C(||H||_{L_\infty} + ||w||_{L_\infty})^p \quad \text{for } s < \delta. \quad (54)$$

Combining (38), (46) and (54) we get

$$\int_{\Omega} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p n^{(1)}}{\phi_\epsilon(x, y)} dS(x) \leq C[||H||_{L_\infty} + ||w||_{L_p} + ||w||_{L_\infty} + ||w_{in}||_{W_p^s(\Gamma_{in})}]^p. \quad (55)$$

The latter gives the bound on the second term on the lhs of (29). However it is convenient to combine it with the estimate on the last term, to which we now move. We have

$$\begin{aligned} & \iint_{\Omega^2} u(x) \frac{\nabla_x |w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy = \\ &= \iint_{\Omega^2} u(x) \nabla_x \left(\frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right) dx dy \iint_{\Omega^2} u(x) |w(x) - w(y)|^p \nabla_x \left(\frac{1}{\phi_\epsilon(x, y)} \right) dx dy \end{aligned} \quad (56)$$

and

$$\begin{aligned} & \iint_{\Omega^2} u(y) \frac{\nabla_y |w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy = \\ &= \iint_{\Omega^2} u(y) \nabla_y \left(\frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right) dx dy - \iint_{\Omega^2} u(y) |w(x) - w(y)|^p \nabla_y \left(\frac{1}{\phi_\epsilon(x, y)} \right) dx dy. \end{aligned} \quad (57)$$

Recalling (31) we see that the terms with $\nabla \left(\frac{1}{\phi} \right)$ cancel. But this time the integrand does not vanish identically and we have to check whether the sum of these integrals makes sense when $\epsilon \rightarrow 0$. This sum equals

$$\iint_{\Omega^2} [u(x) - u(y)] |w(x) - w(y)|^p \frac{(2 + sp) |x - y|^{sp} (x - y)}{(\epsilon + |x - y|^{2+sp})^2} dx dy,$$

and when $\epsilon = 0$ we get

$$(2 + sp) \iint_{\Omega^2} \frac{u(x) - u(y)}{x - y} \frac{|w(x) - w(y)|^p}{|x - y|^{2+sp}} dx dy \leq C \|\nabla u\|_{L_\infty} \|w\|_{W_p^s}^p.$$

Now consider the terms with $\nabla \frac{|w(x) - w(y)|^p}{\phi_\epsilon}$. Combining (33) and (34) with (31) we see that these terms add up to

$$\begin{aligned} & \frac{2}{p} \iint_{\Omega^2} u(x) \nabla_x \left(\frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \right) dx dy = \\ &= - \int_{\Omega} dy \int_{\Omega} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \operatorname{div}_x u(x) dx + \int_{\Omega} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} u \cdot n dS(x). \end{aligned}$$

The first integral is straightforward:

$$\left| \int_{\Omega} dy \int_{\Omega} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} \operatorname{div}_x u(x) dx \right| \leq C(\Omega) \|\nabla u\|_{L_\infty} \|w\|_{W_p^s}^p \quad (58)$$

when $\epsilon \rightarrow 0$ and the boundary integral can be combined with (35) to give

$$\int_{\Omega} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} (n^{(1)} + u \cdot n) dS(x). \quad (59)$$

Now it is useful to consider again $\Omega = \Omega_{in}^r \cup \Omega_{in}^s \cup \Omega \setminus \Omega_{in}$, where the subsets are defined as before. On $\Omega_{in}^r \cup (\Omega \setminus \Omega_{in})$, $n^{(1)}$ is dominating over $u \cdot n$ due to smallness of u . In particular the inner Γ_{out} integral in (59) will be negative and we get analogously to (60)

$$\int_{(\Omega \setminus \Omega_{in}) \cup \Omega_{in}^r} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_{\epsilon}(x, y)} (n^{(1)} + u \cdot n) dS(x) \leq C(\|H\|_{L_{\infty}} + \|w\|_{L_{\infty}} + \|w\|_{L_p})^p \quad (60)$$

for $s < 1$. Concerning the neighbourhood of the singularity points we can estimate similarly to (53)

$$\int_{\Omega_{in}^s} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_{\epsilon}(x, y)} (1 + u^{(1)}) n^{(1)} dS(x) \leq C(\|H\|_{L_{\infty}} + \|w\|_{L_{\infty}})^p \quad (61)$$

for $s < \delta$. But now $n^{(2)}u^{(2)}$ is no longer dominated by $n^{(1)}$ so we have to investigate separately

$$\int_{\Omega_{in}^s} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_{\epsilon}(x, y)} u^{(2)} n^{(2)} dS(x). \quad (62)$$

Notice that we have

$$n(x)|_{\Gamma_{out}} = \frac{[1, -\overline{x_1}'(x_2)]}{\sqrt{1 + [\overline{x_1}'(x_2)]^2}}, \quad n(x)|_{\Gamma_{in}} = \frac{[-1, \underline{x_1}'(x_2)]}{\sqrt{1 + [\underline{x_1}'(x_2)]^2}}. \quad (63)$$

As $dS(x_2) = \sqrt{1 + [\overline{x_1}'(x_2)]^2} dx_2$ on Γ_{out} and $dS(x_2) = \sqrt{1 + [\underline{x_1}'(x_2)]^2} dx_2$ on Γ_{in} , (63) yields

$$dS(x_2)|_{\Gamma_{out}} = \frac{dx_2}{n^{(1)}}, \quad dS(x_2)|_{\Gamma_{in}} = -\frac{dx_2}{n^{(1)}}. \quad (64)$$

and

$$n^{(2)} dS(x_2)|_{\Gamma_{out}} = -\overline{x_1}'(x_2) dx_2, \quad n^{(2)} dS(x_2)|_{\Gamma_{in}} = \underline{x_1}'(x_2) dx_2. \quad (65)$$

Hence we can rewrite (62) as

$$\int_{\Omega} dy \int_0^b \left[\frac{|w(\underline{x_1}) - w(y)|^p}{\phi_{\epsilon}(\underline{x_1}, y)} \underline{x_1}'(x_2) u^{(2)}(\underline{x_1}) - \frac{|w(\overline{x_1}) - w(y)|^p}{\phi_{\epsilon}(\overline{x_1}, y)} \overline{x_1}'(x_2) u^{(2)}(\overline{x_1}) \right] dx_2, \quad (66)$$

where

$$f(\underline{x}_1) := f((\underline{x}_1(x_2), x_2)) \quad \text{and} \quad f(\overline{x}_1) := f((\overline{x}_1(x_2), x_2)).$$

This time both terms in (66) are unsigned and must be treated. We will have to control normal component of the velocity around singularity points and here the assumption (23) will come into play. Let us focus on the Γ_{in} term. By (63), on Γ_{in} we have

$$u \cdot n = -\frac{u^{(1)}}{\sqrt{1 + [\underline{x}_1'(x_2)]^2}} + \frac{u^{(2)}\underline{x}_1'(x_2)}{\sqrt{1 + [\underline{x}_1'(x_2)]^2}}.$$

The first term vanishes around the singularity points, hence (23) on Γ_{in} close to singularity points is equivalent to

$$\frac{|u^{(2)}| |\underline{x}_1'(x_2)|^2}{\sqrt{1 + [\underline{x}_1'(x_2)]^2}} \leq M. \quad (67)$$

On the other hand, we have

$$|u^{(2)}\underline{x}_1'(x_2)| \leq 2 \frac{|u^{(2)}| |\underline{x}_1'(x_2)|^2}{\sqrt{1 + [\underline{x}_1'(x_2)]^2}},$$

which combined with (67) gives boundedness of $\underline{x}_1'(x_2)u^{(2)}(\underline{x}_1)$ in (66). Now the Γ_{in} part of (66) can be dealt with exactly as in the derivation of (53). The Γ_{out} part is treated in the same way integrating this time along the characteristic connecting y with Γ_{out} (notice that we do not use the information on the boundary, we only replace the difference with integral along the characteristic). We conclude

$$\int_{\Omega_{in}^s} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_{\epsilon}(x, y)} u^{(2)} n^{(2)} dS(x) \leq C(\|H\|_{L_{\infty}} + \|w\|_{L_{\infty}})^p \quad \text{for } s < \delta. \quad (68)$$

Combining (60), (61) and (68) we get

$$\begin{aligned} & \int_{\Omega} dy \int_{\Gamma} \frac{|w(x) - w(y)|^p}{\phi_{\epsilon}(x, y)} (n^{(1)} + u \cdot n) dS(x) \leq \\ & \leq C(\|H\|_{L_{\infty}} + \|w\|_{L_{\infty}} + \|w\|_{L_p} + \|w_{in}\|_{W_p^s(\Gamma_{in})})^p + \|\nabla u\|_{L_{\infty}} \|w\|_{W_p^s}^p \end{aligned} \quad (69)$$

Now we are ready to close the estimate (25). To this end we have to recall (29). Taking into account (30)-(36) and (56)-(59), we can replace the lhs of (69) with the integral terms from the lhs of (29), that is, we get

$$\begin{aligned} & \iint_{\Omega^2} \frac{\partial_{x_1}|w(x) - w(y)|^p + \partial_{y_1}|w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy \\ & + \iint_{\Omega^2} \frac{u(x) \cdot \nabla_x |w(x) - w(y)|^p + u(y) \cdot \nabla_y |w(x) - w(y)|^p}{\phi_\epsilon(x, y)} dx dy \leq \\ & \leq C(\|H\|_{L_\infty} + \|w\|_{L_\infty} + \|w\|_{L_p} + \|w_{in}\|_{W_p^s(\Gamma_{in})})^p + \|\nabla u\|_{L_\infty} \|w\|_{W_p^s}^p. \end{aligned}$$

This estimate combined with (29) yields

$$\|w\|_{W_p^s}^p \leq C[\|H\|_{W_p^s} \|w\|_{W_p^s}^{p-1} + \|w\|_{L_\infty}^p + \|w\|_{L_p}^p] + E \|w\|_{W_p^s}^p$$

and applying interpolation inequality (9) we conclude (22). \square

Acknowledgements. This work has been supported by Ideas Plus grant ID 2011 0006 61.

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